# Pseudo-characters

$A$ will be a comm. ring, $R$ an $A$-alg.

For $\rho : R \to M_d(A)$ a rep$^n$, set $\chi_\rho = \mathrm{tr}\circ\rho : R \to A$.

Motivation: We know that traces of rep$^n$ provide a lot of information, even with deformations (Mazur - Carayol)

To what extent can we work with functions $f \in \mathrm{Hom}_A(R,A)$ & still obtain $\chi_\rho$ for some rep$^n$ $\rho$??

Today follows Rouquier's paper (French)

p. characters (Serre, Rouquier) p-representation (Wiles, Taylor, Nyssen)

## §1 A property of $\chi_\rho$

Def$^n$ $f \in \mathrm{Hom}_A(R,A)$ is called central if $f(xy) = f(yx)$ $\forall\, x,y \in R$.

e.g. $\chi_\rho$ is central

Notation ($n^{th}$ signed symmetrisation)

$f$ central, $n \in \mathbb{N}_{>0}$. For $\sigma \in \mathfrak{S}_n$, $\sigma = \sigma_1 \cdots \sigma_m$ prod of disjoint cycles, & $\underline{x} = (x_1,\dots,x_n) \in R^n$

set

$$\underline{x}_{\sigma_i} = x_{i_1} \cdots x_{i_k} \in R, \quad \sigma_i = (i_1,\dots,i_k)$$

$$f_\sigma(\underline{x}) = \prod_{i=1}^m f(\underline{x}_{\sigma_i})$$ (well defined as $f$ central)

Then

$$S_n(f) : R^n \longrightarrow A, \quad \underline{x} \longmapsto \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma)\cdot f_\sigma(\underline{x}) \qquad \varepsilon : \mathfrak{S}_n \xrightarrow{\mathrm{sgn}} \{\pm 1\}$$

Example $n=1$ $S_1(f)(x) = f(x)$

$n=2$ $S_2(f)(x,y) = f(x)f(y) - f(xy)$

Lm $$S_{n+1}(f)(x_1,\dots,x_{n+1}) = \underbrace{f(x_{n+1})\, S_n(f)(x_1,\dots,x_n)} - \sum_{i=1}^n S_n(f)(x_1,\dots,x_{i-1},x_i x_{n+1},x_{i+1},\dots,x_n)$$

Pf $$\sum_{\sigma(n+1)=n+1} \varepsilon(\sigma) f_\sigma(x_1,\dots,x_{n+1}) + \sum_{i=1}^n \sum_{\sigma(i)=n+1} \varepsilon(\sigma) f_\sigma(x_1,\dots,x_{n+1})$$

$$(i,n+1,j,\dots) = (i,j,\dots)(i,n+1)$$ □

$\Rightarrow$ If $S_k(f) \equiv 0$, then $S_n(f) \equiv 0$ $\forall\, n > k$.

Lm $f$ central & assume $n!$ not a zero divisor in $R$.

$$S_n(f) \equiv 0 \iff S_n(f)(x,\dots,x) = 0 \quad \forall\, x \in R.$$

- $A'$ localisation of $A$ at $n!$ & set $R' = A' \otimes_A R$, $f' = \mathrm{id}_{A'} \otimes f$

  $\rightsquigarrow$ replace $f$ by $f'$

  Thus can assume $n!$ is invertable.

- $S^n(R)$ $n^{th}$ Symm space generated by $x^n$, $x \in R$ ($n!$ is invertable)

  $\exists\, g \in \mathrm{Hom}_A(S^n(R), A)$ s.t.

$$\begin{tikzcd} R^n \arrow[r] \arrow[dr, "G", "f"'] & S^n(R) \arrow[d, dashed, "g"] \\ & A \end{tikzcd}$$ (univ. prop.)

□

# Page 2

Tuesday, 22. June 2021 19:27

**Thm** (Frobenius) $\rho: R \to M_d(\mathbb{C})$ rep$^n$. Then $S_n(\chi_\rho) \equiv 0 \quad \forall\, n > d$.

Pf Assume $R = M_d(\mathbb{C})$, $\rho = Id_R$, $\chi_\rho = tr: M_n(\mathbb{C}) \to \mathbb{C}$.

$$S_n(tr)(x, \dots, x) = \sum_{\sigma \in S_n} \varepsilon(\sigma) \cdot \left( \prod_{i=1}^m tr(x^{c_i}) \right) \qquad \sigma = \sigma_1 \cdots \sigma_m, \quad c_i = ord(\sigma_i).$$

This is invariant under conjugation & is continuous.

$\rightsquigarrow$ assume $x$ is diagonal ( diagonlisable matrices are dense in $M_n(\mathbb{C})$)

**Claim** For $W := (\mathbb{C}^d)^{\otimes n}$, then $\prod_{i=1}^m tr(x^{c_i}) = Tr_W(x \cdot \sigma)$

Fix $\{e_1, \dots, e_d\}$ basis of eigenvec. of $x$ on $\mathbb{C}^d$ with $\{\lambda_1, \dots, \lambda_d\}$ corresp. eigenvalues

Basis of $W$ consists of $\underbrace{e_{i_1} \otimes \cdots \otimes e_{i_n}}_{w}$, $i_1, \dots, i_n \in \{1, \dots, d\}$

$x \cdot \sigma(w) = x \cdot e_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes x \cdot e_{i_{\sigma^{-1}(n)}}$

$\Big\{$ contribute to trace

$\iff$ $w$ is fixed by $\sigma$ (i.e. if $\sigma(s) = s'$, then $e_{i_s} = e_{i_{s'}}$)

corresponding eigenvalue is then $\prod_{i=1}^m \lambda_{\sigma_i}^{c_i}$ $\quad \lambda_{\sigma_i} \rightsquigarrow \sigma_i$ eigenvalue.

$$Tr_W(x \cdot \sigma) = \prod_{i=1}^m \underbrace{\left( \sum_{j=1}^d \lambda_j^{c_i} \right)}_{tr(x^{c_i})}$$

$S_n(tr)(x, \dots, x) = Tr_W\left( x \cdot \sum \varepsilon(\sigma) \cdot \sigma \right)$

but $n > d$, $\left( \sum \varepsilon(\sigma) \cdot \sigma \right) \cdot W = 0$

in $w$ $\exists\, s, s'$ s.t. $e_{i_s} = e_{i_{s'}}$

$$\left( \sum \varepsilon(\sigma) \sigma \right) \cdot w = \sum_{\sigma \in A_n} \sigma \cdot w - \sum_{\sigma \in A_n} \sigma \cdot (s, s') \cdot w = 0 \qquad \square$$

**Rem** In Rouquier he proved this for any rep$^n$ $\rho: R \to \Gamma$, $\Gamma$ an Azumaya Algebra/A
" generalized notion of central simple over $A$ "

local rings $\rightsquigarrow$ $\Gamma \otimes_A A/m$ is central simple over $A/m$.

The proof still takes $R = \Gamma$ & then argues that $\Gamma$ can be taken as $M_d(A)$, $A$ field of char 0.

# §2: Pseudo-characters

"form a converse to the previous theorem"

Def$^n$ $f$ central is called a pseudo-character if $\exists\, k>0$ s.t. $S_{k+1}(f)=0$

(many authors include $k!$ is invertable in $A$)

The degree of $f$ is the smallest $d$ s.t. $S_{d+1}(f)=0$.

Prop If $A$ local, $d!$ is invertable & $\deg(f)=d$, then $f(1)=d$.

Pf $0=S_{d+1}(f)(1,\dots,1)=(f(1)-d)\,S_d(f)(1,\dots,1)=\dots=(f(1)-d)(f(1)-(d-1))\cdots f(1)$.

each difference $i-i'$ is invertable so at most one $f(1)-i$ lies in max$^l$ ideal & all others are invertable. (+)

$\Rightarrow f(1)\in\{0,1,2,\dots,d\}$

If $f(1)<d$, then $0=S_{d+1}(f)(x_1,\dots,x_d,1)=\underset{\text{invertable}}{(f(1)-d)}\,\underbrace{S_d(f)(x_1,\dots,x_d)}_{\neq 0}$ ⚡. $\square$

Cor In this setup with $\rho: R\to M_d(A)$ a rep$^n$: $\deg(\chi_\rho)=\chi_\rho(1)=\dim(\rho)$.

Def$^n$ - $f$ a p-char. $\ker(f):=\{x\in R \mid f(xy)=0\ \ \forall y\in R\}$

$f$ is called faithful if $\ker(f)=0$

- If $d!$ is invertable in $A$ & $\deg(f)=d$, the characteristic polynomial of $x$ for $f$ is given by $P_{x,f}(X)=X^d+\sum_{i=1}^{d}\frac{(-1)^i}{i!}S_i(f)(x,\dots,x)X^{d-i}\in A[X]$

example - $d=1$, $P_{x,f}(X)=X-f(x)$

$d=2$ $P_{x,f}(X)=X^2-f(x)X+\frac{f(x)^2-f(x^2)}{2}$

- If $f=\chi_\rho$, then $P_{x,f}$ is the characteristic poly. of $\rho(x)$.

Rem As $f$ is central, $\ker(f)$ will be a 2-sided ideal.

$\rightsquigarrow\ \bar f: R/\ker(f)\longrightarrow A$ is a faithful p-ch.

Prop (Cayley-Hamilton) Let $f$ be a p-ch, of deg. $d$, with $d!$ invertable in $A$.

If $f$ is faithful, then $P_{x,f}(x)=0\quad\forall x\in R$

(show $S_{d+1}(f)(x,\dots,x,y)=(-1)^d\cdot d!\, f(P_{x,f}(x)\cdot y)$)

As we are thinking about central simple algebras, we will have to look at idempotents.

Lm $e\in R$ be an idempotent.

I) $f_e:=f|_{eRe}$ is a p-ch (from def$^n$)

II) If $A$ local with $d!$ invertable, then $f(e)\in\{0,1,2,\dots,d\}$ (replace 1 with $e$ in (+))

III) —— " —— & $f$ faithful, then

- $f(e)\neq 0$ ($f(e^n)=0\ \forall n \Rightarrow P_{e,f}(X)=X^d \Rightarrow e^d=0$)
- $f_e$ faithful ($x\in\ker(f_e)$, $r\in R$: $\underset{\overset{\shortparallel}{0}}{f(x.r)}=\underset{\overset{\shortparallel}{f(x.(ere))}\ =\ 0}{f(x.r.e)}+\underset{\overset{\shortparallel}{f((1-e)x.r)}\ =\ 0}{f(x.r.(1-e))}$)
- $R$ has at most $d$ non-zero pairwise orthogonal idemp.

$k\le f(e_1+\dots+e_k)\le d$

# Page 4

Tuesday, 22. June 2021 19:59

## §3 : pseudo-characters over a field

**Thm** Let $f$ be a p-ch of $R/k$ with $\deg < \mathrm{char}(k)$. Then $R/\ker(f)$ is a semi-simple algebra.

If $k$ is separably closed, then $f = \chi_\rho$ for some rep$^n$ $\rho: R \to M_d(k)$, $d = \deg(f)$.

**Pf** Assume $f$ faithful ($P_{x,f}(x) = 0$)

- $V$ simple $R$-mod, $D = \mathrm{End}_R(V)$ (division ring). Then $\dim_D(V) \leq d$:
  - $R \twoheadrightarrow M_n(D^{op})$ $\forall\, n \leq \dim(V)$

    (Density thm : $X \subset V$ fin, $D$-lin indep set, $M \in \mathrm{End}_R(V)$. $\exists\, r \in R$ s.t. $r.x = Ax$ $\forall\, x \in X$)
  - each elt of $R$ is annihilated by a poly. of deg $d$, not true in $M_n(D^{op})$, $n > d$!
- $R$ has a fin # of simple modules up to isom:

  $V_1, \dots, V_k$ distinct non-isom simples. $\rho_i : R \to \mathrm{End}_{D_i}(V_i) = M_{n_i}(D_i^{op})$ $n_i = \dim_{D_i}(V_i)$
  - $\bigoplus \rho_i : R \twoheadrightarrow \bigoplus \mathrm{End}_{D_i}(V_i) \cong \bigoplus d_i V_i$ ( $\mathrm{Im}(R) = \bigoplus r_i V_i$ & $\rho_i$ surj $\Rightarrow r_i = d_i$ . )
  - $\exists$ $e_i$ idem of $R$ s.t. $\rho(e_i) = 1_{\mathrm{End}_{D_i}(V_i)}$

    $k$-subalg. generated by any pre-image of $1_{\mathrm{End}_{D_i}(V_i)}$ in $R$ is f.d. ($P_{x,f}$ deg $d$)
    apply [Bourbaki] to get $e_i$.
  - Moreover $e_1, \dots, e_k$ are pairwise orthogonal, distinct idem.

  $\Rightarrow k \leq d$.
- $J(R) = 0$

  $x \in J(R)$, $x$ nilpotent :

  $$0 = P_{x,f}(x) = a \cdot x^c \underbrace{(1 + x\,Q(x))}_{\text{inv.}} \Rightarrow a \cdot x^c = 0 \Rightarrow x^c = 0 \ \& \ c > 0.$$

  $x^{2^i} \neq 0$, $x^{2^{i+1}} = 0$ ($\Rightarrow f(x^{2^{i+1}}) = 0$)

  Then
  $$0 = S_{d+1}(f)(x^{2^i}, \dots, x^{2^i}) = f(x^{2^i})\, S_d(f)(x^{2^i}, \dots, x^{2^i}) = \vdots = f(x^{2^i})^{d+1} = 0$$

  repeating $\Rightarrow f(x) = 0$

  $\forall\, y \in R$, $f(xy) = 0 \Rightarrow x \in \ker(f) = 0$.

$$\Rightarrow R = \prod_{i=1}^{k} M_{d_i}(D_i^{op}) \quad \text{i.e. semi-simple.}$$

If $k$ is separably closed, then $D_i = k$.

$$f(x) = \sum_{i,j} f(e_i x e_j) = \sum_i f(e_i x e_i) = \sum_i f_{e_i}(e_i x e_i)$$

WTS: p-ch of $e_i R e_i = M_{d_i}(k)$ is the trace of a rep$^n$ :

$E_{i,j} = E_{i,i} E_{i,j} - E_{i,j} E_{i,i} \Rightarrow f(E_{i,j}) = 0$

$E_{i,i} - E_{j,j} = E_{i,j} E_{j,i} - E_{j,i} E_{i,j} \Rightarrow f(E_{i,i}) = f(E_{j,j})$

$\rightsquigarrow f(x) = c \cdot \mathrm{tr}(x)$

$f(E_{i,i}) \in \{1, \dots, d\} \Rightarrow f$ is a sum of $f(E_{1,1})$ copies of standard rep$^n$.

$\square$.

# Page 5

Tuesday, 22. June 2021 20:20

Rem : $\exists$ notion of absolutely irreducible for pseudo-characters & in the previous theorem $\rightsquigarrow$ $R/_{\ker(f)} \cong M_n(k)$.

Thm (Nyssen, Rouquier)

Suppose $A$ is a local strictly Henselian ring
(Hensel lemma hold + residue field $k$ is separable closed)
$f$ pseudo char $R \longrightarrow A$ of deg $d$ s.t. $\bar{f} := f \otimes \mathbb{1} : R \otimes k \longrightarrow k$ is an irred p. char.
Then $R/_{\ker(f)} \cong M_d(A)$ & $f = \chi_\rho$ for $\rho : R \longrightarrow R/_{\ker(f)}$

Idea

- $R/_{J(R)} \cong \dfrac{R \otimes k}{\ker(\bar{f})} \cong M_n(k)$

- Combine Hensel with the lemma of Bourbaki to lift idem from $R/J$ to $R$

- require additional results related to p-ch & tensor products.

# Page 6

Tuesday, 22. June 2021 20:28

## §4 Deformations (Kisin)

$A \in AF_{W(\mathbb{F})}$ "fin local Artinian $W(\mathbb{F})$-alg". $A/m_A = \mathbb{F}$.

$\Gamma_n(A) = \ker\left( Gl_n(A) \xrightarrow{\pi} Gl_n(\mathbb{F}) \right)$

$G$ pro-fin gp, $V_{\mathbb{F}}$ a $\mathbb{F}[G]$-module

fix basis & consider $\rho : G \to Gl_n(A)$ s.t. $\bar{\rho} = \pi \circ \rho$

$D_{V_{\mathbb{F}}}(A) = \{$ isom. of deformations $V_{\mathbb{F}}$ to $A \} = D^{\square}_{V_{\mathbb{F}}}(A) / \Gamma_n(A)$-conj

$V_A$ free $A$-mod

$\iota : V_A \otimes \mathbb{F} \to V_{\mathbb{F}}$ "G-equiv."

**Def<sup></sup>** - $f_{\mathbb{F}} : G \to \mathbb{F}$ is a p-ch if its lin ext $\tilde{f} : \mathbb{F}[G] \to \mathbb{F}$ is a p-ch.

- $D_{f_{\mathbb{F}}}(A) = \{ f_A : G \to A \text{ p-ch} \mid \pi \circ f_A = f_{\mathbb{F}} \}$

**Thm** (Nyssen, Rouquier)

If $G$ satisfies $\Phi_p$ & $\bar{\rho} : G \to Gl_n(\mathbb{F})$ is ab. irred, then

$$D_{V_{\mathbb{F}}} \xrightarrow{\simeq} D_{\chi_{\bar{\rho}}} \quad \text{as functors on } AR_{W(\mathbb{F})}$$

Saw framed deformations are pro-representable in Ying-Ying's talk.

**Prop** If $G$ satisfies $\Phi_p$ & $f_{\mathbb{F}} : G \to \mathbb{F}$ is a p-ch. Then $D_{f_{\mathbb{F}}}$ is pro-rep<sup></sup> by a complete local Noetherian $W(\mathbb{F})$-alg.

**Idea**: "immediate if $G$ was a top. fin. gen gp."

- $\ker(f) := \{ g \in G \mid f(gh) = f(h) \ \forall g \in G \}$.

$\tilde{f}((g-1)\cdot h) = f(gh) - f(h)$. $\rightsquigarrow$ $g \in \ker(f) \iff g-1 \in \ker(\tilde{f})$.

- $A \in AR_{W(\mathbb{F})}$ & $H \subseteq \ker(f_{\mathbb{F}})$ s.t. quot is a max pro-p gp. Then $H \subset \ker(f_A)$

$\rightsquigarrow$ $f_A$ can be take as a p-ch of $G/H$.

- $\exists$ fin subset $S \subset G$ s.t. $f_A$ determined by its restriction to $S$

$\rightsquigarrow$ $G/H$ will be topologically fin. gen

# Page 7

Tuesday, 22. June 2021 20:44

## Connection to Groupoids

Saw that $D_{V_{\mathbb{F}}}(A)$ forms a groupoid over $AR_{W(\mathbb{F})}$.

"captures geom. of deformation theory more accurately than its functor of iso. cl".

$f_{\mathbb{F}} : G \longrightarrow \mathbb{F}$ p-ch

$Aug_{W(\mathbb{F})} := \{ (A, B) \mid A \in AR_{W(\mathbb{F})}, B \text{ A-alg} \}$ category topology $B$ induced from $m_A$.

Groupoid on $Aug_{W(\mathbb{F})}$ :

$Rep_{f_{\mathbb{F}}}(B)$ = Cat. fin free $B$-modules $V_B$ with cont. $G$ action s.t.

$$tr\left( g|_{V_B \otimes_A A/m_A} \right) = f_{\mathbb{F}}(g) \quad \forall g \in G$$

$Rep^{\square}_{f_{\mathbb{F}}}(B) \rightarrow$ Category $(V_B, \beta_B)$ for $\beta_B$ a fixed basis.

Then $D_{f_{\mathbb{F}}}(A, B) := \varprojlim_{\substack{A' \subset B \\ \in AR_{W(\mathbb{F})}}} Rep_{f_{\mathbb{F}}}(A')$

" $=$ p-ch $f_B : G \longrightarrow B$ s.t. $f_B \otimes_B B/m_A B = f_{\mathbb{F}}$ "

<u>Lm</u> If $G$ satisfies $\Phi_p$, then $Rep^{\square}_{f_{\mathbb{F}}}$ is representable by a formal scheme over $Spf\ R_{f_{\mathbb{F}}}$ which is formally of fin. type.

"left as an exercise in Kisin"